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Design of a Longitudinal Ride-Control System by Zakian's Method of Inequalities

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In this paper, Zakian's method of inequalities is applied to the design of a ride-control system for a STOL aircraft. The purpose of the controller is to reduce the normal acceleration experienced by both passengers and crew. The method is based on the synthesis of a controller such that a set of performance specifications and constraints is satisfied. Controllers are designed on the basis of the characteristics of both the closed-loop step response and the closed-loop error response. It is shown that the design of a single-input, single-output controller by the method of inequalities is straightforward, and can be achieved by using a sequence of formulations until the designer is satisfied that no further improvement is necessary.

Nomenclature

A = plant matrix
a_y = lateral acceleration
a_z = normal acceleration
$a_z(x,t)$ = normal acceleration at station x
b = input vector
C = comfort rating
c_i = constraint or bound
$D_1(s), D_2(s)$ = denominators of $\hat{G}_{i1}(s)$, $\hat{G}_{i2}(s)$
$d(t)$ = input disturbance
$e(t)$ = error signal
E = disturbance matrix
$f(t)$ = combined input function
g = acceleration due to gravity
$G(s)$ = open-loop transfer function
$G_i(s)$ = open-loop aproper transfer function
$\hat{G}_i(s)$ = open-loop proper transfer function
$\hat{G}_{i1}(s)$ = open-loop transfer function for fast mode
$\hat{G}_{i2}(s)$ = open-loop transfer function for slow mode
i = integer
I_y = pitch inertia
j = integer
$\hat{K}_i$ = real number
$\hat{K}_{\max}$ = maximum external disturbance
$K(s,p)$ = single-input, single-output controller
$L(s)$ = transfer function of power control and actuator
$\mathcal{L}\, z(t)$ = Laplace transforms $= \bar{z}(s)$
$m_1^T(x), m_2^T$ = measurement row vectors
MBP = moving boundaries process
$M_u, M_w, M_{\dot{w}}, M_q, M_\eta$ = aerodynamic derivatives $= \partial M/\partial u$, $\partial M/\partial w$, $\partial M/\partial \dot{w}$, $\partial M/\partial q$, $\partial M/\partial \eta$, respectively
$N_i(s)$ = numerator of $\hat{G}_i(s)$
$N_{i1}(s), N_{i2}(s)$ = numerators of $\hat{G}_{i1}(s)$, $\hat{G}_{i2}(s)$
p_i = controller parameter
p = controller parameter vector
q = integer
$q(t)$ = pitch rate
r = integer
$r(t)$ = external input
s = Laplace operator
t = time
t_s = settling period
$u_g(t)$ = atmospheric disturbance vector
$u(t)$ = velocity component in x direction
u_e = initial velocity component in x direction
$u_g(t)$ = gust velocity component in x direction
$U_{\min}, U_{\max}$ = minimal, maximal controller outputs
V = undisturbed flight speed
$V_{\min}, V_{\max}$ = minimal, maximal rates of change of controller output
$w(t)$ = velocity component in z direction
$w_g(t)$ = gust velocity component in z direction
x = distance of point ahead of center of gravity
$x(t)$ = longitudinal state vector
$X_u, X_w, X_{\dot{w}}, X_q, X_\eta$ = aerodynamic derivatives $= \partial X/\partial u$, $\partial X/\partial w$, $\partial X/\partial \dot{w}$, $\partial X/\partial q$, $\partial X/\partial \eta$, respectively
$y(t)$ = system response
$\hat{y}(t)$ = system step response
$Z_u, Z_w, Z_{\dot{w}}, Z_q, Z_\eta$ = aerodynamic derivatives $= \partial Z/\partial u$, $\partial Z/\partial w$, $\partial Z/\partial \dot{w}$, $\partial Z/\partial q$, $\partial Z/\partial \eta$, respectively
α = positive stability specification
$\Delta(s)$ = characteristic polynomial
ϵ = prescribed error level
$\eta(t)$ = elevator angle
$\eta_D(t)$ = controller output demand
λ_j = closed-loop pole
Θ_e = initial pitch angle
$\theta(t)$ = perturbation in pitch angle

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$\mu(t)$ = controller output
ν = number of closed-loop poles
σ = negative abscissa of stability
σ_{a_y} = root-mean-square value of lateral acceleration
σ_{a_z} = root-mean-square value of normal acceleration
ϕ_i = inequality function
ω_1 = damped frequency of short-period oscillation
ω_2 = damped frequency of long-period oscillation

I. Introduction

As a result of increasing public demand for short-haul air transportation, short takeoff and landing (STOL) aircraft have been found to be a good solution in terms of economy, noise reduction, and traffic relief. It is well known (see, for example, Ref. 1) that the wing loading must be decreased in order to decrease both the takeoff and landing speeds. However, wing loading reduction introduces certain operational deficiencies, such as sensitivity to gust loads.[2] In order to suppress the effects of atmospheric turbulence and to overcome the associated high level of work load required from the pilot, it is generally necessary to install a ride quality control system on a STOL transport aircraft. Here a ride quality control system is taken to refer to an automatic control system, the purpose of which is to reduce the accelerations experienced both by passengers and crew. A number of ride comfort models have been proposed by several researchers (see, for example, Refs. 3-5).

One of the common ride quality criteria is the Ride Comfort Rating.[6] Indeed, Wolf, Rezek, and Gee[7] defined Ride Comfort Rating using, as a basis, a five-point rating scale. Their model was based on passengers' opinions of ride comfort obtained from experiments using the NASA Jetstar airborne simulator. For cases when the normal acceleration, a_z, is at least 1.6 times greater than the lateral acceleration, a_y (as was the case with the Jetstar aircraft), the comfort rating, C, is given by the expression

$$C=2+7.6\sigma_{a_y}+11.9\sigma_{a_z} \tag{1}$$

where σ_{a_y} and σ_{a_z}, respectively, are the root-mean-square values (expressed in units of g) of the lateral and normal accelerations of the aircraft. Thus it is a principal objective of a ride quality control system to suppress the values of lateral and longitudinal acceleration in order that the comfort rating can be brought down to acceptable and pleasant levels.

In a thesis, Saoullis[3] developed a mathematical model of an executive jet STOL aircraft. This model is used in this paper as the basis for the design of a ride quality control system with the objective of improving the normal acceleration response under turbulent conditions. The design is undertaken by applying Zakian's method of inequalities (see, for example, Refs. 8-21). A brief review of this method is presented in Sec. II of this paper, and an appropriate mathematical model is developed in Secs. III and IV. Two approaches to the design are presented in Sec. V: first, there is a design based on a reduced-order model, and second, there is a design based on a more complete model. Appropriate conclusions are drawn in Sec. VI.

II. The Method of Inequalities

A. General

Detailed accounts of the method of inequalities and its applications to the design of linear, time-invariant control systems have been given by Zakian and Al-Naib.[8,9] For the sake of completeness, the method is reviewed briefly in this section. Although the method can be applied to the design of controllers for multi-input, multi-output systems, in this paper the method is applied to the design of a single controller, $K(s,p)$, for the class of single-input, single-output system depicted in Fig. 1. This controller has the general structure

$$K(s,p)=\frac{p_1(1+p_2s)\dots(1+p_rs)}{s^q(1+p_{r+1}s)\dots(1+p_ns)} \tag{2}$$

Although any proper rational function could be used for the controller $K(s,p)$, the form Eq. (2) chosen here restricts $K(s,p)$ to real poles and zeros. This has certain advantages in the application considered in this paper. First, the designer formulates the problem in terms of a number of inequalities in the form

$$\phi_i(p)\le c_i \qquad i=1,2,\dots \tag{3}$$

where c_i is a real number, p denotes the real vector $(p_1, p_2,\dots,p_n)$ representing the parameters of the controller which is being designed. For each p, there is $\phi_i(p)$ which is a real scalar and which represents an aspect of the dynamical behavior of the system under design: these aspects include steady-state error, the rise time, the overshoot, the settling time, the undershoot, and the maximal and minimal controller output.

The real number c_i in Eq. (3) represents the numerical bound on the particular aspect of dynamical behavior described by $\phi_i(p)$, where the particular selection of the function ϕ_i depends upon the chosen design criteria. In formulating the problem, the designer has to choose the functions, ϕ_i, the bounds, c_i, and the structure, $K(s,p)$, of the controller. Preference is given, of course, to controllers with the simplest structures. Once this is done, the problem has been formulated, and it only remains to compute a value of the vector p (if one exists) which satisfies all the inequalities expressed by Eq. (3). This value of p is the numerical solution to the problem thus formulated and its elements determine the precise form of the controller structure. By examining the result, the designer may decide to refine the formulation of the problem. This may be done by adding new inequalities, or by removing some, or by replacing some inequalities with others, or by choosing a different structure of controller. In this way, the designer goes through a sequence of formulations until satisfied that no further improvement is necessary.

A simple algorithm, called the moving boundaries process (MBP) is available[8] for obtaining the numerical solution to the set of inequalities. The MBP algorithm is simple and effective and has been used extensively in practice,[11-22] although an analysis of its convergence properties has yet to be made. Another algorithm,[23-26] which is more complicated, but with known convergence properties, has recently been proposed for solving such functional inequalities.

To ensure a known level of asymptotic stability for the closed-loop system, another inequality is added. In the case of a single-input, single-output system, a negative abscissa of stability, σ (which can be computed by repeated use of the Routh array[27]), ensures an adequately stable closed-loop system. Thus the additional inequality is

$$\sigma\le-\alpha \tag{4}$$

where

$$\sigma=\max\{\text{real}(\lambda j)\} \qquad j=1,2,\dots,\nu \tag{5}$$

and the λ_j are the ν poles of the closed-loop transfer function. The specification of stability is then expressed by the inequality

$$\alpha<0 \tag{6}$$

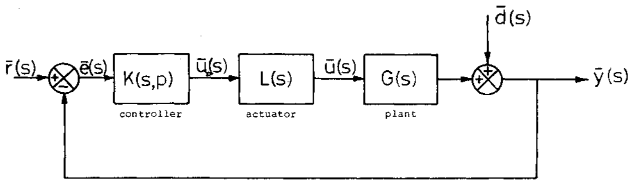

Fig. 1 Single-input, single-output closed-loop control system.

If the initial selection of p gives rise to an unstable closed-loop system, then the MBP method is used to find a stable point by satisfying the inequality Eq. (4). With this as the starting point, the MBP method is used to solve all the inequalities simultaneously. Each trial of the MBP method involves a test for stability followed by the computation of the $\phi_i(p)$ which are obtained from the system step response $\hat{y}(t)$ for the class of system shown in Fig. 1, with $\bar{d}(s)=0$, or the error response $e(t)$ for the class of system shown in Fig. 1, with $\bar{d}(s)\neq 0$. Computation of these responses can be carried out effectively by means of I_{MN} approximants.[28,29]

B. Measures of Performance

This subsection presents an outline of recent work by Zakian[9,10] in which the practice of specifying control system performance through the step response is justified. The closed-loop system specifications are formulated as inequalities in order to permit the design of a controller by use of the method of inequalities. The block diagram of a standard single-input, single-output feedback control system is shown in Fig. 1, and the corresponding closed-loop transfer function, is given by the expression

$$\bar{w}(s)=\frac{\bar{y}(s)}{\bar{r}(s)}=\frac{K(s,p)L(s)G(s)}{1+K(s,p)L(s)G(s)} \tag{7}$$

where $K(s,p)$ is a controller with the structure given in Eq. (2). The closed-loop unit step response, $\hat{y}(t)$, is defined by

$$\hat{y}(t)=\mathcal{L}^{-1}[\bar{w}(s)/s] \tag{8}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operation.

In the case depicted in Fig. 1 where an equivalent disturbance, $d(t)$, is introduced to the system output, $y(t)$, the closed-loop system can be treated as though it were subjected to a combined input function, $f(t)$, given by the equation

$$f(t)=r(t)-d(t) \tag{9}$$

It follows from Eq. (9) that the corresponding error signal, $e(t)$, when $r(t)$ is a unit step is given by the expression

$$e(t)=[1-\hat{y}(t)]f(0)+\int_0^t[1-\hat{y}(\tau)]f^{(1)}(t-\tau)\mathrm{d}\tau \tag{10}$$

One of the aims of the design is to achieve a small error, $e(t)$, which can be achieved by reducing the size of $|1-\hat{y}(t)|$ to a sufficiently small and acceptable range of values. This can be done using the method of inequalities having applied Holder's inequality[8] to Eq. (10), and with the use of conventional step-response characteristics such as the steady-state error, the rise time, the overshoot, and the settling time. These step-response characteristics of the system are defined in classical control literature (for example, Refs. 30,31). When the input function, $r(t)$, is one of a known class, then certain other characteristics of the class have to be known.[9,32] In this situation, the design objective is to construct a controller such that the error settles down to within a prescribed level, ϵ, in a prescribed period of time called the settling period, t_s. In other words,

$$\sup|e(t)|\leq\epsilon \qquad t\geq t_s \tag{11}$$

C. Control Constraints

The design method illustrated in this paper is intended to improve the output step-response characteristics, and to improve the error response when the system is subjected to disturbances. However, the mathematical model of the aircraft developed in Secs. III and IV is conditionally linear.[9] In other words, the linearity of the model is only preserved under the condition that the controller output, $\mu(t)$, is limited in magnitude to the maximum deflection angle which the actuating element is allowed to produce. In addition, the rate of change, $\mathrm{d}\mu/\mathrm{d}t$, of controller output is limited by the maximum rate of change which can be produced by the actuating element. Thus, in addition to the constraints determined by the requirements for desirable transient-response characteristics, there are two other constraints dictated by the system construction. By ensuring that neither of these constraints is violated, it will be possible to have greater confidence in the design of a controller which is based on a linearized model. In terms of inequalities, these constraints can be expressed as

$$\inf\{\mu:t\geq 0\} \quad \geq \quad U_{\min}$$

$$\sup\{\mu:t\geq 0\} \quad \leq \quad U_{\max} \tag{12}$$

and

$$\inf\left\{\frac{\mathrm{d}\mu}{\mathrm{d}t}:t\geq 0\right\} \quad \geq \quad V_{\min}$$

$$\sup\left\{\frac{\mathrm{d}\mu}{\mathrm{d}t}:t\geq 0\right\} \quad \leq \quad V_{\max} \tag{13}$$

where $U_{\min}$, $U_{\max}$ and $V_{\min}$, $V_{\max}$ are the minimal and maximal controller outputs and the minimal and maximal rates of change of the controller output, respectively.

III. Mathematical Model

A. Equations of Motion

The equations of longitudinal motion of an aircraft can be expressed in a number of ways using a variety of notations and nomenclature. For the purpose of illustrating the method of inequalities, it will be assumed that the aircraft behaves as a rigid body and that the motion after a small disturbance from 1g straight-and-level flight consists only of small perturbations in the response variables. Thus the longitudinal motion will be modeled by the following dimensional, linearized, ordinary differential equations:

Forward translation,

$$m[\dot{u}(t)+w_e q(t)]=X_u[u(t)+u_g(t)]+X_w[w(t)+w_g(t)]$$

$$+X_{\dot{w}}\dot{w}(t)+X_q q(t)+X_\eta\eta(t)-mg\cos\Theta_e\theta(t) \tag{14a}$$

Vertical translation,

$$m[\dot{w}(t)-u_e q(t)]=Z_u[u(t)+u_g(t)]+Z_w[w(t)+w_g(t)]$$

$$+Z_{\dot{w}}\dot{w}(t)+Z_q q(t)+Z_\eta\eta(t)-mg\sin\Theta_e\theta(t) \tag{14b}$$

Pitching rotation,

$$I_y\dot{q}(t)=M_u[u(t)+u_g(t)]+M_w[w(t)+w_g(t)]$$

$$+M_{\dot{w}}\dot{w}(t)+M_q q(t)+M_\eta\eta(t) \tag{14c}$$

Pitching kinematics,

$$\dot{\theta}(t)=q(t) \tag{14d}$$

In designing a longitudinal ride-control system, an expression for the normal acceleration, $a_z(x,t)$, at a point distance x ahead of the center of gravity is required. This expression is

$$a_z(x,t) = \dot{w}(t) - u_e q(t) - x\dot{q}(t) \quad (15)$$

Equations (14) can be written in the usual state-space form as

$$\dot{x}(t) = Ax(t) + b\eta(t) + Eu_g(t) \quad (16)$$

where the longitudinal state vector, $x(t)$, is

$$x(t) = \begin{bmatrix} u(t) \\ w(t) \\ q(t) \\ \theta(t) \end{bmatrix} \quad (17a)$$

and where the atmospheric disturbance vector, $u_g(t)$, is

$$u_g(t) = \begin{bmatrix} u_g(t) \\ w_g(t) \end{bmatrix} \quad (17b)$$

The expression for the normal acceleration, $a_z(x,t)$, given in Eq. (15) can now be written in terms of the vectors $\dot{x}(t)$ and $x(t)$ in the form

$$a_z(x,t) = m_1^T(x)\dot{x}(t) + m_2^T x(t) \quad (18)$$

where the measurement row vectors, $m_1^T(x)$ and m_2^T, are given by the respective equations

$$m_1^T(x) = [0,1,-x,0] \quad (19a)$$

and

$$m_2^T = [0,0,-V,0] \quad (19b)$$

It follows from Eqs. (16) and (18) that the normal acceleration, $a_z(x,t)$, can be expressed in terms of the longitudinal state vector, $x(t)$, the control input, $\eta(t)$, and the atmospheric disturbance vector, $u_g(t)$. Thus

$$a_z(x,t) = [m_1^T(x)A + m_2^T]x(t) + m_1^T(x)b\eta(t) + m_1^T(x)Eu_g(t) \quad (20)$$

The open-loop transfer functions for the normal acceleration, $a_z(x,t)$, can be derived directly from Eq. (18) or (20).

IV. Illustrative Example

A. Open-Loop Model

The method of inequalities is illustrated in this paper by considering the appropriate longitudinal data for a twin-engined executive jet aircraft flying at low speed. This model has been used by a number of researchers (see, for example, Ref. 3).

At a flight speed of 40 m/s in the approach configuration, the longitudinal rigid-body dynamics are described by the following matrices:

Plant matrix,

$$A = \begin{bmatrix} -0.0166 & +0.108 & 0 & -9.81 \\ -0.175 & -1.01 & +40.0 & 0 \\ +0.00131 & -0.00991 & -0.546 & 0 \\ 0 & 0 & +1.0 & 0 \end{bmatrix} \quad (21a)$$

Input vector,

$$b = \begin{bmatrix} +1.97 \\ -17.20 \\ -2.26 \\ 0 \end{bmatrix} \quad (21b)$$

Disturbance matrix,

$$E = \begin{bmatrix} -0.166 & +0.108 \\ -0.175 & -1.01 \\ +0.00131 & -0.00991 \\ 0 & 0 \end{bmatrix} \quad (21c)$$

Measurement row vector,

$$m_1^T(x) = [0,1,0,0] \quad (21d)$$

Measurement row vector,

$$m_2^T = [0,0,-40.0,0] \quad (21e)$$

The appropriate open-loop transfer functions used in the design of an elevator controller to suppress the normal acceleration response to both horizontal and vertical gusts are given by the expressions

$$\bar{a}_z(0,s)/\bar{\eta}(s) = G_1(s) = (-17.2s^4 - 10.021s^3 + 84.142s^2 - 1.133s)/\Delta(s) \quad (22a)$$

$$\bar{a}_z(0,s)/\bar{u}_g(s) = G_2(s) = (-0.175s^4 - 0.0956s^3 - 0.1223s^2)/\Delta(s) \quad (22b)$$

$$\bar{a}_z(0,s)/\bar{w}_g(s) = G_3(s) = (-1.01s^4 - 0.587s^3 - 0.0195s^2 - 0.0300s)/\Delta(s) \quad (22c)$$

where the characteristic polynomial, $\Delta(s)$, is

$$\Delta(s) = s^4 + 1.573s^3 + 0.995s^2 + 0.033s + 0.0300 \quad (23)$$

Each of the three open-loop transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$ has aproper form: that is, the numerator has the same order as the denominator. An alternative formulation in terms of proper transfer functions, $\hat{G}_1(s)$, $\hat{G}_2(s)$, and $\hat{G}_3(s)$, is as follows:

$$G_1(s) = \hat{K}_1 + \hat{G}_1(s) = -17.2 + (17.03s^3 + 101.26s^2 - 0.562s + 0.516)/\Delta(s) \quad (24a)$$

$$G_2(s) = \hat{K}_2 + \hat{G}_2(s) = -0.175 + (0.180s^3 + 0.052s^2 + 0.0058s + 0.0052)/\Delta(s) \quad (24b)$$

$$G_3(s) = \hat{K}_3 + \hat{G}_3(s) = -1.01 + (1.001s^3 + 0.986s^2 + 0.0036s + 0.0303)/\Delta(s) \quad (24c)$$

The block-diagrams of the closed-loop descriptions of the systems in terms of both the aproper and proper open-loop transfer functions are presented in Figs. 2 and 3, respectively. In this paper, the proper form is used in order to facilitate the inclusion of step disturbances.

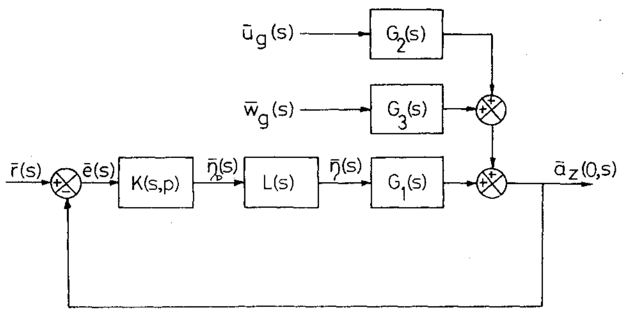

Fig. 2 Closed-loop ride-control system with aproper transfer functions.

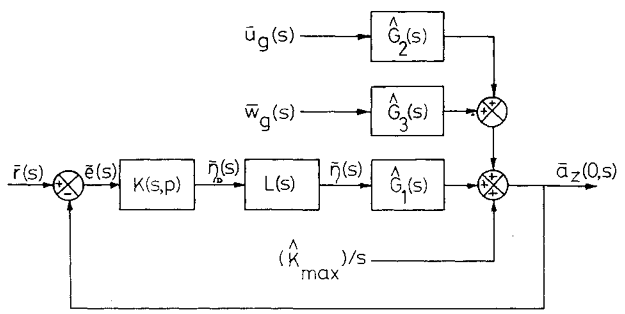

Fig. 3 Closed-loop ride-control system with proper transfer functions and maximum disturbance.

Before proceeding with the design analysis, the poles and zeros of the open-loop transfer functions $\hat{G}_1(s)$, $\hat{G}_2(s)$, and $\hat{G}_3(s)$ are determined. These are as follows:

Poles of $\Delta(s)$,

$$-0.794 \pm i0.597 \tag{25a}$$

$$+0.008 \pm i0.174 \tag{25b}$$

Zeros of $\hat{G}_1(s)$,

$$+0.003 \pm i0.071 \tag{26a}$$

$$-5.953 \tag{26b}$$

Zeros of $\hat{G}_2(s)$,

$$+0.053 \pm i0.267 \tag{27a}$$

$$-0.394 \tag{27b}$$

Zeros of $\hat{G}_3(s)$,

$$+0.013 \pm i0.172 \tag{28a}$$

$$-1.011 \tag{28b}$$

It is evident from Eqs. (25) that the longitudinal modes of response comprise the classical short-period oscillation and the long-period Lanchester phugoid. The respective damped frequencies of these two modes are

$$\omega_1 = 0.597 \quad \text{rad/s} \tag{29a}$$

$$\omega_2 = 0.174 \quad \text{rad/s} \tag{29b}$$

which are sufficiently well separated to allow the model to be simplified in the first instance before designing the structure, $K(s,p)$, of the controller. Once this structure has been determined, it is then a simple matter to refine it using the original unsimplified model.

There are a number of methods of simplification (for example, Refs. 33-36), of which the following are commonly used in the analysis of problems in aircraft dynamics:

1) Cancellation of the poles and zeros which correspond to the long-period oscillation.

2) Recomputation of the transfer functions of a reduced-order system by neglecting the forward-force equation, changes in forward speed, and the kinematic equation.

3) Decomposition of the open-loop transfer functions into two independent fast and slow subsystems, each of second order.

In this paper, a type 3 simplification based on a partial-fraction, transfer-function decomposition is performed.

Thus, if

$$\hat{G}_i(s) = \frac{N_i(s)}{D_1(s)D_2(s)} \qquad i = 1,2,3 \tag{30a}$$

then

$$\hat{G}_i(s) = \hat{G}_{i1}(s) + \hat{G}_{i2}(s) = \frac{N_{i1}(s)}{D_1(s)} + \frac{N_{i2}(s)}{D_2(s)} \tag{30b}$$

The appropriate decompositions for $\hat{G}_1(s)$, $\hat{G}_2(s)$, and $\hat{G}_3(s)$, corresponding to the data in Eqs. (24-29) are as follows:

$$\hat{G}_1(s) = \hat{G}_{11}(s) + \hat{G}_{12}(s)$$

$$= \frac{12.531(s-7.719)}{s^2+1.587s+0.986} + \frac{4.497(s+0.547)}{s^2-0.015s+0.030} \tag{31a}$$

$$\hat{G}_2(s) = \hat{G}_{21}(s) + \hat{G}_{22}(s)$$

$$= \frac{0.184(s-0.314)}{s^2+1.587s+0.986} - \frac{0.00458(s-0.772)}{s^2-0.015s+0.030} \tag{31b}$$

$$\hat{G}_3(s) = \hat{G}_{31}(s) + \hat{G}_{32}(s)$$

$$= \frac{1.012(s-1.004)}{s^2+1.587s+0.986} - \frac{0.0109(s-0.057)}{s^2-0.015s-0.030} \tag{31c}$$

The block diagram of the reduced-order, closed-loop system is obtained directly from Fig. 3 by replacing $\hat{G}_1(s)$, $\hat{G}_2(s)$, and $\hat{G}_3(s)$ with $\hat{G}_{11}(s)$, $\hat{G}_{21}(s)$, and $\hat{G}_{31}(s)$, respectively. In this figure

$$\hat{K}_{\max} = \sup\{\pm\hat{K}_1\eta_{\max} \pm \hat{K}_2 u_{g_{\max}} \pm \hat{K}_3 w_{g_{\max}}\} \tag{32}$$

B. Performance Specification and Constraints

In this paper the normal acceleration response of an aircraft subjected to step longitudinal gusts, u_g and w_g, is considered since ride quality improvement is the only concern. In actual aircraft control system design, the acceleration response of the vehicle to control inputs would also be considered in order to realize satisfactory handling qualities. For ride quality improvement, only the following inequalities are used in the design process.

1) Performance specification for step response:

Steady-state error in normal acceleration,

$$\phi_1 = 0 \tag{33a}$$

Rise time,

$$\phi_2 \leq 1.5 \text{ s} \tag{33b}$$

Overshoot,

$$\phi_3 \leq 10\% \tag{33c}$$

Settling time,

$$\phi_4 \leq 5 \text{ s} \tag{33d}$$

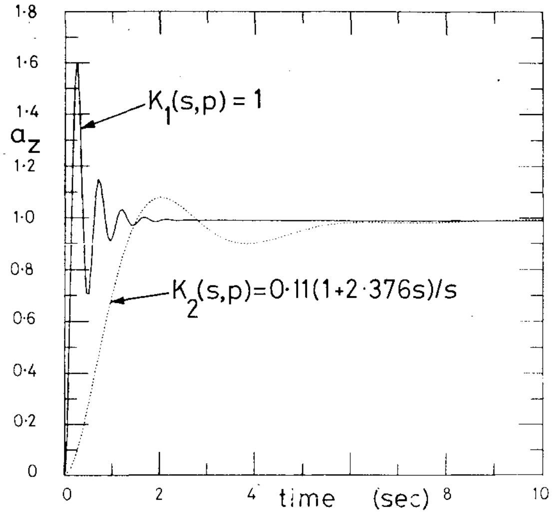

Fig. 4 Step responses of reduced-order model: controllers $K_1(s,p)$ and $K_2(s,p)$.

2) Performance specification for error response:

Settling period, t_s,

$$\phi_5 \leq 5 \text{ s} \tag{34a}$$

Sup $|e(t)|$, $t \geq t_s$,

$$\phi_6 \leq 0.3 \, g \tag{34b}$$

3) Elevator constraints:

The maximum deflection angle of the elevator actuator is limited to 0.4 rad with a maximum velocity of 0.436 rad/s, and it is possible to apply these bounds to the controller output, η_D. This is valid because the response of the actuator $L(s)$ is much faster than that of the remainder of the system. Thus

$$\inf\{\eta_D : t \geq 0\} = \phi_7 \qquad \phi_7 \geq -0.4 \text{ rad}$$
$$\{\eta_D : t \geq 0\} = \phi_8 \qquad \phi_8 \leq -0.4 \text{ rad} \tag{35a}$$

$$\inf\left\{\frac{d\eta_D}{dt} : t \geq 0\right\} = \phi_9 \qquad \phi_9 \geq -0.436 \text{ rad/s}$$
$$\sup\left\{\frac{d\eta_D}{dt} : t \geq 0\right\} = \phi_{10} \qquad \phi_{10} \leq -0.436 \text{ rad/s} \tag{35b}$$

V. Design

A. General

The design of a ride quality control system using Zakian's method of inequalities is undertaken in this paper in two stages. First, a reduced-order model of the type developed in Sec. IV is used. This is done to get some feel for the structure of the controller, $K(s,p)$, which is satisfactory, before moving on to use the complete model in the second stage. In both stages of the design process, the open-loop transfer function of the power control and actuator is assumed to be

$$L(s) = \frac{\bar{\eta}(s)}{\bar{\eta}_D(s)} = \left(\frac{12.5}{s+12.5}\right)\left(\frac{2500}{s^2+37.5s+2500}\right) \tag{36}$$

The position and velocity constraints of the power control and actuator are given by the inequalities described by Eqs. (35).

B. Reduced-Order Model

1. Unit Controller

The unit controller closed-loop step response of the reduced-order model described by Eqs. (31) is shown in Fig. 4. In this instance, the controller has the structure

$$K_1(s,p) = 1 \tag{37}$$

The values of the appropriate functionals are presented in Table 1, column 1.

It is evident from the values of the appropriate functionals that the overshoot value of 63% is unacceptable. In addition, the elevator control angle varies between −0.6 and +1.0 rad, both of which exceed the permitted linear range of the actuating elements. Thus it is necessary to design a controller transfer function, $K(s,p)$, the inclusion of which will enable the step-response characteristics of the closed-loop system to satisfy the set of inequalities listed in Sec. IV.B. as inequalities (33-35). In this section, two different formulations of the controller are described for the reduced-order model and three further formulations of the controller are described in Sec. V.C. for the complete model.

2. Proportional Controller

A proportional controller has the structure

$$K(s,p) = p_1 \tag{38a}$$

so that the parameter vector, p, in Eq. (2) has the form

$$p = [p_1] \tag{38b}$$

In order to apply Zakian's method of inequalities, a number of different starting values of p_1 were used. After 40 iterations of the moving boundaries process, no solution to the design problem could be found for any of the arbitrarily chosen starting values of p_1. A conflict between settling time and overshoot was also noted.

3. Proportional-plus-Integral Controller

Recognizing that a zero-valued, steady-state error is required, the proportional-plus-integral controller was chosen

Table 1 System functionals for different controllers

	Functional \ Controller	Step response $K_1(s,p)$	Step response $K_2(s,p)$	Error response $K_2(s,p)$	Step response $K_3(s,p)$	Error response $K_4(s,p)$
Steady-state error	$\phi_1 = 0g$	0.01	0	...	0	...
Rise time	$\phi_2 \leq 1.5$ s	0.12	1.46	...	1.94	...
Overshoot	$\phi_3 \leq 10\%$	63	8	...	0.9	...
Settling time	$\phi_4 \leq 5$ s	1.46	4.90	...	2.99	...
Settling period, t_s	$\phi_5 \leq 5$ s	...	...	4.6	...	4.0
sup $\|e(t)\|$, $t \geq t_s$	$\phi_6 \leq 0.3\,g$	...	...	0.16	...	0.30
U_{min}	$\phi_7 \geq -0.4$ rad	−0.61	+0.01	+0.02	−0.01	−0.31
U_{max}	$\phi_8 \leq +0.4$ rad	+1.00	+0.02	+0.18	+0.39	+0.39
V_{min}	$\phi_9 \geq -0.436$ rad/s	−4.83	−0.003	−0.155	−0.169	−0.017
V_{max}	$\phi_{10} \leq +0.436$ rad/s	+2.61	+0.020	+0.033	+0.004	+0.148

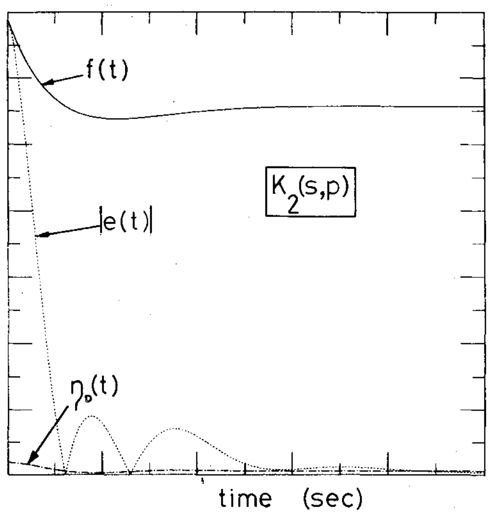

Fig. 5 Error response of reduced-order model: controller $K_2(s,p)$.

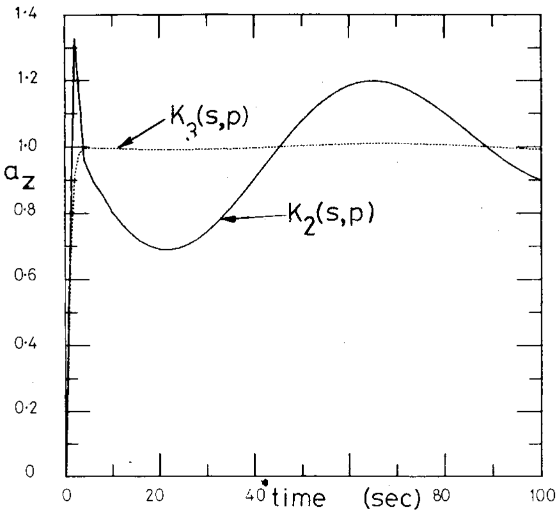

Fig. 6 Step response of complete model: controllers $K_2(s,p)$ and $K_3(s,p)$.

to have the structure

$$K_2(s,p) = p_1(1+p_2 s)/s \tag{39a}$$

so that the parameter vector, p, in Eq. (2) has the form

$$p = [p_1, p_2]^T \tag{39b}$$

With the initial value

$$p = [1, 1]^T \tag{39c}$$

an approximate solution of the set of inequalities was found after 12 iterations of the moving boundaries process. The resulting value of the parameter vector was

$$p = [0.011, 2.376]^T \tag{39d}$$

so that the corresponding controller has the form

$$K_2(s,p) = 0.011(1+2.376s)/s \tag{40}$$

The corresponding response of the reduced-order model to a unit step input is shown in Fig. 4, and the corresponding values of the appropriate functionals are presented in Table 1, column 2. Thus the controller is satisfactory with respect to all the selected performance specifications and constraints.

When the aircraft is subjected to step disturbances of 5 m/s in forward speed, u_g, and 1 m/s in vertical gust speed, w_g, the error response described in Sec. II.B. is used as a further design criterion. In this case, the behavior of the error response to a combined input function of the form described in Eq. (9) should satisfy the set of inequalities expressed by Eqs. (34). Rather than apply the method of inequalities again, the controller described by Eq. (40) proved to be generally satisfactory. The corresponding error response of the reduced-order model is shown in Fig. 5, and the corresponding values of the appropriate functionals are presented in Table 1, column 3.

C. Complete Model

1. *Proportional-plus-Integral Controller*

The controller described by Eq. (40) proved to be satisfactory in all respects in the case of both the step response and the error response of the reduced-order model. In designing a controller for the complete model, the same controller is tried in the first instance. The corresponding step response is shown in Fig. 6, from which it can be seen that there is evidence of a lightly damped, long-period oscillation, as might have been expected. One hundred iterations of the moving boundaries process with different starting controller parameters did not yield satisfaction of the specified inequalities. At this stage, it was decided to increase the complexity of the controller structure in order to compensate the system for the effects of the long-period oscillation. To achieve this objective, a phase-lead compensator should be added with a break point at the frequency of the long-period oscillation.

2. *Proportional-plus-Integral Controller with Phase Lead*

Thus the complexity of the controller is increased, and the effects of using a proportional-plus-integral controller with phase lead are investigated.

Such a controller has the structure

$$K_3(s,p) = \frac{p_1(1+p_2 s)(1+p_3 s)}{s(1+p_4 s)} \tag{41}$$

so that the parameter vector, p, in Eq. (2) has the form

$$p = [p_1, p_2, p_3, p_4]^T \tag{42}$$

Initial values of p_1, p_2 are taken to be equal to the corresponding values given in Eq. (39d) for the reduced-order model. Initial values of p_3 and p_4 for the phase-lead compensator are chosen to be 10 and 1, respectively.

Thus with the initial value

$$p = [0.011, 2.376, 10, 1]^T \tag{43}$$

an approximate solution to the set of inequalities was found after 25 iterations of the moving boundaries process. The resulting value of the parameter vector was

$$p = [0.400, 0.655, 7.000, 4.600]^T \tag{44}$$

The corresponding response of the complete model to a unit step input is shown in Fig. 6, and the corresponding values of the appropriate functionals are presented in Table 1, column 4. Thus the controller is generally satisfactory with respect to all the selected performance specifications and constraints, with the exception of the rise time. However, it can readily be shown, in the case of the error response when the aircraft is subjected to the step disturbances, $u_g = 5$ m/s and $w_g = 1$ m/s, that the initial value of controller demand is unacceptably high. There are two techniques which are normally used in such a situation. First, the design constraints have to be relaxed, or second, the complexity of the controller has to be increased.

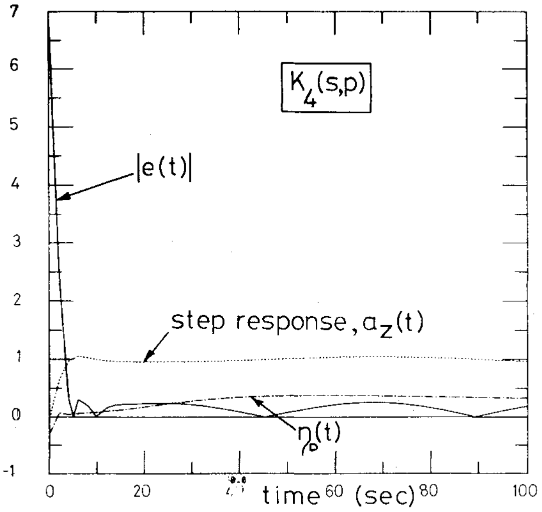

Fig. 7 Step and error response of complete model: controller $K_4(s,p)$.

3. Proportional-plus-Integral Controller with Double Phase Lead

In this numerical example, it was decided that the design constraints should not be relaxed in the first instance. Thus the complexity of the controller is increased further, and the effects of using a proportional-plus-integral controller with double phase lead are investigated. Such a controller has the structure

$$K_4(s,p) = \frac{p_1(1+p_2s)(1+p_3s)(1+p_4s)}{s(1+p_5s)(1+p_6s)} \tag{45}$$

so that the parameter vector, p, in Eq. (2) has the form

$$p = [p_1,p_2,p_3,p_4,p_5,p_6]^T \tag{46}$$

With the initial value

$$p = [0.01,0.89,7,1,4.6,1]^T \tag{47}$$

an approximate solution[8] to the set of inequalities was found after 40 iterations of the moving boundaries process. The resulting value of the parameter vector was

$$p = [0.00436,5.305,8.422,-0.498,19.102,0.114]^T \tag{48}$$

The corresponding response to a unit step input and the error response to a combined input function are shown in Fig. 7. The values of the appropriate functionals are presented in Table 1, column 5. Thus the controller is satisfactory with respect to all the performance specifications and constraints, and can be regarded as being acceptable.

VI. Conclusion

Almost all of the recent trends in control system design have been toward the use of numerical procedures, and the application of the method of inequalities to control system design is another manifestation of this trend. In this paper, the design of a longitudinal ride-control system for a short takeoff and landing aircraft has been undertaken by using the method of inequalities.

Design of conditionally linear, time-invariant, control systems by the method of inequalities comprises two main criteria: 1) step-response criterion of design (overshoot, rise time, settling time,...,etc., defined in Ref. 8); and 2) error-response criterion of design ($\sup |e(t)| \leq \epsilon$, $t \geq t_s$).

The results obtained by using the step-response criterion are not enough to ensure the satisfactory operation of the system when it is subjected to external disturbances. In such cases, the error-response criterion proves to be useful.

It is evident that controller design by the method of inequalities results in a simple configuration for the controller for the closed-loop system. The design comprises a single input and a single output, and no additional states need to be observed or measured. The controller structure resulting from using the method of inequalities is not necessarily unique, which is often an advantage rather than a disadvantage in many cases, since it gives the manufacturer more than one solution from which a controller can be selected using other criteria.

References

[1] Stinton, D., *The Anatomy of the Aeroplane,* G.T. Foulis and Co. Ltd., Yeovil, England, 1966, pp. 41-44, 152-154.

[2] Hoak, D.E. et al., "USAF Stability and Control Datcom," Flight Control Division, Air Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, 1968.

[3] Saoullis, S.L.A., "An Optimal Ride Control System for an Executive Jet Aircraft," M.Sc. Thesis, Loughborough University of Technology, U.K., 1980.

[4] Taylor, J., *Manual on Aircraft Loads,* Pergamon Press, New York, 1965, pp. 191-202, 205-218.

[5] Jacobson, I.D. and Kuhlathau, A.R., "Determining STOL Ride Quality Criteria—Passenger Acceptance," *Journal of Aircraft,* Vol. 10, March 1973, pp. 163-166.

[6] Erkelens, L.J.J. and Schuring, J., "Investigation on a Passenger Ride-Comfort Improvement System with Limited Control Surface Actuator Performance for a Flexible Aircraft," NLR TR75 140U, 1975.

[7] Wolf, T.D., Rezek, J.W., and Gee, S.W., "Passenger Ride Quality Response to an Airborne Simulator Environment," NASA TM-X-3295, 1975.

[8] Zakian, V. and Al-Naib, U.M.T., "Design of Dynamical and Control Systems by the Method of Inequalities," *Proceedings of the IEE,* Vol. 120, Nov. 1973, pp. 1421-1427.

[9] Zakian, V., "New Formulation for the Method of Inequalities," *Proceedings of the IEE,* Vol. 126, June 1979, pp. 579-584.

[10] Zakian, V., "The Performance and Sensitivity of Classical Control Systems," *International Journal of Systems Science,* Vol. 9, March 1978, pp. 343-355.

[11] Taiwo, O., "Improvement of Turboalternator Response by the Method of Inequalities," *International Journal of Control,* Vol. 27, Feb. 1978, pp. 305-311.

[12] Taiwo, O., "On the Design of Feedback Controllers for the Continuous Stirred Tank Reactor by the Method of Inequalities," *Chemical Engineering Journal,* Vol. 17, Issue 1, 1979, pp. 3-12.

[13] Taiwo, O., "Design of a Multivariable Controller of a High-Order Turbofan Engine Model by Zakian's Method of Inequalities," *IEE Transactions AC-23,* Oct. 1978, pp. 926-928.

[14] Taiwo, O., "Design of Multivariable Controller for an Advanced Turbofan Engine by Zakian's Method of Inequalities," *Transactions of ASME, Journal of Dynamic Systems, Measurement and Control,* Vol. 101, Dec. 1979, pp. 299-307.

[15] Taiwo, O., "Application of the Method of Inequalities to the Multivariable Control of Binary Distillation Column," *Chemical Engineering Science,* Vol. 35, Issue 4, 1980, pp. 847-858.

[16] Coelho, C.A.D., "Compensation of the Speed Governor of a Water Turbine by the Method of Inequalities," *Transactions of ASME, Journal of Dynamic Systems, Measurement and Control,* Vol. 101, Sept. 1979, pp. 205-211.

[17] Gray, J.O. and Al-Janabi, T.H., "Towards the Numerical Design of Non-linear Feedback Systems by Zakian's Method of Inequalities," *Proceedings of the IFAC Symposium,* Udine, Italy, 1976, pp. 327-334.

[18] Gray, J.O. and Al-Janabi, T.H., "The Numerical Design of Feedback Control Systems Containing a Saturation Element by the Method of Inequalities," *Proceedings of the 7th IFIP Conference on Optimization Techniques,* Vol. 2, 1975, pp. 510-521.

[19] Dahshan, A.M., "A Comparison between the Method of Inequalities and Other Techniques in Control Systems Design," *International Journal of Systems Science,* Vol. 12, No. 9, 1981, pp. 1149-1156.

[20] Dahshan, A.M., "Application of the Method of Inequalities to the Control Design of Systems with Dead Time," Control Systems Centre Rept. 500, University of Manchester Institute of Science and Technology, U.K., 1981.

[21] Dahshan, A.M., "Alternatives in the Design of Control Systems by the Method of Inequalities," Control Systems Centre Rept. 503, University of Manchester Institute of Science and Technology, U.K., 1981.

[22]Dahshan, A.M., "Application of Zakian's Method of Inequalities in Control System Design," M.Sc. Dissertation, University of Manchester Institute of Science and Technology, U.K., 1979.

[23]Gonzaga, C., Polak, E., and Trahan, R., "An Improved Algorithm for Optimization Problems with Functional Inequality Constraints," *IEEE Transactions on Automatic Control,* Vol. AC-25, Feb. 1980, pp. 49-54.

[24]Polak, E., Trahan, R., and Mayne, D.Q., "Combined Phase I, Phase II Methods of Feasible Directions," *Mathematical Programming,* Vol. 17, Jan. 1979, pp. 61-74.

[25]Becker, H.G., Heunis, A.J., and Mayne, D.Q., "Computer-Aided Design of Control Systems via Optimization," *Proceedings of the IEE,* Vol. 126, June 1979, pp. 573-578.

[26]Polak, E., "Algorithms for a Class of Computer-Aided Design Problems, A Review," *Automatica, IFAC,* Vol. 15, No. 5, Sept. 1979, pp. 531-538.

[27]Zakian, V., "Computation of Abscissa of Stability by Repeated Use of the Routh Array," *IEEE Transactions AC-24,* 1979, pp. 604-607.

[28]Zakian, V., "Properties of I_{MN} and J_{MN} Approximants and Application to Numerical Inversion of Laplace Transforms and Initial Value Problems," *Journal of Mathematical Analysis and Applications,* Vol. 50, April 1975, pp. 191-222.

[29]Zakian, V. and Edwards, M.J., "Tabulation of Constants for Full Grade I_{MN} Approximants," *Math. of Comp.,* Vol. 32, No. 142, 1978, pp. 519-531.

[30]Ogata, K., *Modern Control Engineering,* Prentice-Hall, New Jersey, 1975.

[31]Power, H.M. and Simpson, R.J., *Introduction to Dynamics and Control,* McGraw Hill, New York, 1978.

[32]Coelho, C.A.D., "Design of Robust Control Systems by the Method of Inequalities," Ph.D. Thesis, University of Manchester Institute of Science and Technology, U.K., 1980.

[33]Chemouil, P., "Analysis and Control of Multi-Time Scale Systems," Control Systems Centre Rept. 470, University of Manchester Institute of Science and Technology, U.K., Jan. 1980.

[34]Porter, B., "Singular Perturbation Method in the Design of Stabilizing State Feedback Controllers for Multivariable Linear Systems," *International Journal of Control,* Vol. 20, No. 4, 1974, pp. 693-695..

[35]Kokotovic, P.V. and Haddad, A.H., "Controllability and Time-Optimal Control of Systems with Slow and Fast Modes," *IEEE Transactions AC-20,* Feb.1975, pp. 111-113.

[36]Chow, J.H. and Kokotovic, P.V., "A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Modes," *IEEE Transactions AC-21,* Oct. 1976, pp. 701-705.